Newton's Formulae for Interpolation

P. Sam Johnson

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Overview

The concept of interpolation is the selection of a function $p(x)$ from a given class of functions in such a way that the graph of

$$y = p(x)$$

passes through a finite set of given data points.

We restrict the **interpolating function** $p(x)$ to being a polynomial.

Polynomial interpolation theory has a number of important uses. Its primary uses is to furnish some mathematical tools that are used in developing methods in the areas of approximation theory, numerical integration, and the numerical solution of differential equations.

Newton's forward and backward formulae for interpolation are discussed.

Errors in Polynomial Interpolation

Let the function $y(x)$, defined by the $(n+1)$ points

$$(x_i, y_i) \qquad (i = 0, 1, 2, \ldots, n)$$

be continuous and differentiable $(n+1)$ times, and let $y(x)$ be approximated by a polynomial $\phi_n(x)$ of degree not exceeding n such that

$$\phi_n(x_i) = y_i$$

for $i = 0, 1, 2, \ldots, n$.

Using the polynomial $\phi_n(x)$ of degree n, we can obtain approximate values of $y(x)$ at some points other than x_i $(0 \le i \le n)$.

Since the expression $y(x) - \phi_n(x)$ vanishes for $x = x_0, x_1, \ldots, x_n$ we put

$$y(x) - \phi_n(x) = L\pi_{n+1}(x) \tag{1}$$

where

$$\pi_{n+1}(x) = (x - x_0)(x - x_1) \cdots (x - x_n)$$

and L is to be determined such that the equation (1) holds for some x' in (x_0, x_n). Clearly

$$L = \frac{y(x) - \phi_n(x)}{\pi_{n+1}(x)}.$$

We construct a function $F(x)$ such that

$$F(x) = y(x) - \phi_n(x) - L\pi_{n+1}(x) \tag{2}$$

where L is given as above.

It is clear that

$$F(x_0) = F(x_1) = \cdots = F(x_n) = F(x') = 0$$

that is, $F(x)$ vanishes $(n+2)$ times in the interval $x_0 \leq x \leq x_n$.

By the repeated application of Rolle's theorem, $F^{(n+1)}(x)$ must vanish in the interval $[x_0, x_n]$ at some point ξ.

On differentiating $F(x) = y(x) - \phi_n(x) - L\pi_{n+1}(x)$, $(n+1)$ times with respect to x, we obtain

$$0 = F^{(n+1)}(\xi) = y^{(n+1)}(\xi) - L(n+1)!$$

so that

$$L = \frac{y^{(n+1)}(\xi)}{(n+1)!} \text{ for some } \xi \in (x_0, x_n).$$

Hence

$$y(x) - \phi_n(x) = \frac{y^{(n+1)}(\xi)}{(n+1)!}\pi_{n+1}(x) \quad \text{for some } \xi \in (x_0, x_n).$$

We thus obtain the required expression for the error

$$y(x) - \phi_n(x) = \frac{\pi_{n+1}(x)}{(n+1)!}y^{(n+1)}(\xi) \quad \text{for some } \xi \in (x_0, x_n). \tag{3}$$

Since $y(x)$ is, generally, unknown and hence we do not have any information concerning $y^{(n+1)}(x)$, formula (3) is **almost useless in practical computations**.

On the other hand, **it is extremely useful in theroetical work in different branches of numerical analysis**.

Finite Differences

Assume that we have a table of values

$$(x_i, y_i) \qquad (i = 0, 1, 2, \ldots, n)$$

of any function $y = f(x)$, the values of x being equally spaced h.

Suppose that we are required to recover the values of $f(x)$ for some intermediate values of x, or to obtain the derivative of $f(x)$ for some x in the range $x_0 \leq x \leq x_n$.

The methods for the solution to these problems **are based on the concept of the 'differences' of a function** which we now proceed to define.

Forward Differences

If $y_0, y_1, y_2, \ldots, y_n$ denote a set of values of y, then

$$y_1 - y_0, \quad y_2 - y_1, \quad \ldots, \quad y_n - y_{n-1}$$

are called the **differences of** y. Denoting these differences by

$$\Delta y_0, \ \Delta y_1, \ \ldots, \ \Delta y_{n-1}$$

respectively, we have

$$\Delta y_0 = y_1 - y_0, \quad \Delta y_1 = y_2 - y_1, \ldots, \Delta y_{n-1} = y_n - y_{n-1}$$

where Δ is called **forward difference operator** and $\Delta y_0, \Delta y_1 \ldots$, are called **first forward differences**. The differences of the first forward differences are called **second forward differences** and are denoted by $\Delta^2 y_0, \Delta^2 y_1, \ldots$. Similarly, one can define **third forward differences, fourth forward differences**, etc.

Backward and Central Differences

The differences

$$y_1 - y_0, \quad y_2 - y_1, \quad \ldots, \quad y_n - y_{n-1}$$

are called **first backward differences** if they are denoted by

$$\nabla y_1, \quad \nabla y_2, \quad \cdots, \quad \nabla y_n$$

respectively, so that $\nabla y_1 = y_1 - y_0, \ \nabla y_2 = y_2 - y_1, \ \ldots, \ \nabla y_n = y_n - y_{n-1}$, where ∇ is called the **backward difference operator**.

The **central difference operator** δ is defined by the relations

$$y_1 - y_0 = \delta y_{1/2}, \quad y_2 - y_1 = \delta y_{3/2}, \ldots y_n - y_{n-1} = \delta y_{n-1/2}.$$

Similarly, higher-order central difference can be defined.

Shift and Averaging Operators

We can observe the following

$$\Delta y_0 = \nabla y_1 = \delta y_{1/2}, \quad \Delta^3 y_2 = \nabla^3 y_5 = \delta^3 y_{7/2}, \ldots.$$

Difference formulae can easily be established by symbolic methods, using the **shift** operator E, and the **averaging** (or, **mean**) operator μ, in addition to the operators, Δ, ∇ and δ already defined.

The **shift operator** E is defined by the equation

$$Ey_r = y_{r+1}.$$

The **averaging operator** μ is defined by the equation

$$\mu y_r = \frac{1}{2}\left(y_{r+\frac{1}{2}} + y_{r-\frac{1}{2}}\right).$$

Symbolic Relations and Separation of Symbols

From the definitions, the following relations can easily be established:

$$\begin{aligned} \nabla &= 1 - E^{-1} \\ \delta &= E^{1/2} - E^{-1/2} \\ \mu &= \frac{(E^{1/2} + E^{-1/2})}{2} \\ \mu^2 &= 1 + (1/4)\delta^2 \\ \Delta &= \nabla E = \delta E^{1/2} \\ E &\equiv e^{hD}. \end{aligned}$$

Newton's Formulae For Interpolation

Given the set of $(n+1)$ values,

$$(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n),$$

of x and y, it is required to find $y_n(x)$, a polynomial of the nth degree such that y and $y_n(x)$ agree at the tabulated points. Let the values of x be equidistant, i.e. let

$$x_i = x_0 + ih, \quad i = 0, 1, 2, \ldots.$$

Since $y_n(x)$ is a polynomial of the nth degree, it may be written as

$$\begin{aligned} y_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots \\ \ldots + a_n(x - x_0)(x - x_1) \ldots (x - x_{n-1}). \end{aligned}$$

Imposing the condition that y and $y_n(x)$ should agree at the set of tabulated points, we obtain

$$
\begin{aligned}
a_0 &= y_0 \\
a_1 &= \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h} \\
a_2 &= \frac{\Delta^2 y_0}{h^2 2!} \\
a_3 &= \frac{\Delta^3 y_0}{h^3 3!} \\
&\vdots \\
a_n &= \frac{\Delta^n y_0}{h^n n!}.
\end{aligned}
$$

Setting $x = x_0 + ph$ and substituting for $a_0, a_1, \ldots, a_n$, the above equation becomes

$$y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots \\ \cdots + \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}\Delta^n y_0$$

which is **Newton's forward difference interpolation formula** and is useful for interpolation **near the beginning** of a set of tabular values.

Difference Table

The values $\boxed{\text{inside the boxes}}$ of the following difference table are used in deriving the Newton's forward difference interpolation formula.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	$\boxed{y_0}$					
		$\boxed{\Delta y_0}$				
x_1	y_1		$\boxed{\Delta^2 y_0}$			
		Δy_1		$\boxed{\Delta^3 y_0}$		
x_2	y_2		$\Delta^2 y_1$		$\boxed{\Delta^4 y_0}$	
		Δy_2		$\Delta^3 y_1$		$\boxed{\Delta^5 y_0}$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$
						$\vdots$
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_n	y_n					

To find the error committed in replacing the function $y(x)$ by means of the polynomial $y_n(x)$, we obtain

$$y(x) - y_n(x) = \frac{(x - x_0)(x - x_1) \cdots (x - x_n)}{(n+1)!} y^{(n+1)}(\xi)$$

for some $\xi \in (x_0, x_n)$.

As remarked earlier we do not have any information concerning $y^{(n+1)}(x)$, and therefore the above formula is useless in practice.

Neverthless, if $y^{(n+1)}(x)$ does not vary too rapidly in the interval, a useful estimate of the derivative can be obtained in the following way. Expanding $y(x+h)$ by Taylor's series, we obtain

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2!} y''(x) + \cdots$$

Neglecting the terms containing h^2 and higher powers of h, this gives

$$y'(x) \approx \frac{1}{h}[y(x+h) - y(x)] = \frac{1}{h}\Delta y(x)$$

Writing $y'(x)$ as $Dy(x)$ where $D \equiv d/dx$, the **differentiation operator**, the above equation gives the operator relations

$$D \equiv \frac{1}{h}\Delta \text{ and so } D^{n+1} \equiv \frac{1}{h^{n+1}}\Delta^{n+1}.$$

We thus obtain

$$y^{(n+1)}(x) \approx \frac{1}{h^{n+1}}\Delta^{n+1}y(x)$$

The equation can therefore be written as

$$y(x) - y_n(x) = \frac{p(p-1)(p-2)\cdots(p-n)}{(n+1)!}\Delta^{n+1}y(\xi)$$

for some $\xi \in (x_0, x_n)$, which is suitable for computation.

Newton's Backward Interpolation Formula

Suppose we assume $y_n(x)$ in the form

$$y_n(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + \cdots$$
$$\cdots + a_n(x - x_n)(x - x_{n-1}) \ldots (x - x_1)$$

and then impose the condition that y and $y_n(x)$ should agree at the tabulated points $x_n, x_{n-1}, \ldots, x_2, x_1, x_0$, we obtain (after some simplification)

$$y_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \cdots + \frac{p(p+1)\cdots(p+n-1)}{n!}\nabla^n y_n$$

where $p = (x - x_n)/h$.

This is **Newton's backward difference interpolation formula** and ite uses tabular values to the left of y_n. This formula is therefore useful for interpolation **near the end of** the tabular values.

Difference Table

The values inside the boxes of the following difference table are used in deriving the Newton's backward difference interpolation formula.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_0	y_0					
$\vdots$	$\vdots$					
		∇y_{n-4}				
x_{n-4}	y_{n-4}		$\nabla^2 y_{n-3}$			
		∇y_{n-3}		$\nabla^3 y_{n-2}$		
x_{n-3}	y_{n-3}		$\nabla^2 y_{n-2}$		$\nabla^4 y_{n-1}$	
		∇y_{n-2}		$\nabla^3 y_{n-1}$		$\boxed{\nabla^5 y_n}$
x_{n-2}	y_{n-2}		$\nabla^2 y_{n-1}$		$\boxed{\nabla^4 y_n}$	
		∇y_{n-1}		$\boxed{\nabla^3 y_n}$		
x_{n-1}	y_{n-1}		$\boxed{\nabla^2 y_n}$			
		$\boxed{\nabla y_n}$				
x_n	$\boxed{y_n}$					

It can be shown that the error in Newton's backward difference formula may be written as

$$y(x) - y_n(x) = \frac{p(p+1)(p+2)\cdots(p+n)}{(n+1)!} h^{n+1} y^{(n+1)}(\xi)$$

where $x_0 < \xi < x_n$ and $x = x_n + ph$.

References

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